Supplementary material for the article:

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Supplementary Material

1 MEAN-FIELD MODEL

We will here give more details on the derivation of the equation (6). First, let us rewrite the equation of motion (5):

$$dx_i = y_i dt, \tag{S1a}$$

$$dy_i = F(v)dt - F(v + y_i)dt + \sum_{\substack{j=1\\j\neq i}}^{N} \kappa(x_{\tau,j} - x_i)dt + \sqrt{2d}\, dw_i. \tag{S1b}$$

Since $F(v)$ is cubic in v, its Taylor series terminates with a cubic term

$$F(v + y_i) = F(v) + F'(v)y_i + \frac{1}{2}F''(v)y_i^2 + \frac{1}{6}F'''(v)y_i^3, \tag{S2}$$

yielding

$$dx_i = y_i dt, \tag{S3a}$$

$$dy_i = -((3av^2 - 2bv + c)y_i + (3av - b)y_i^2 + ay_i^3)dt + \sum_{\substack{j=1\\j\neq i}}^{N} \kappa(x_{\tau,j} - x_i)dt + \sqrt{2d}\, dw_i. \tag{S3b}$$

We will now perform the cumulant analysis of this system of globally coupled units in the thermodynamical limit of an infinitely large $N \to \infty$ ensemble. Our system now becomes

$$dx_i = y_i dt, \tag{S4a}$$

$$dy_i = -((3av^2 - 2bv + c)y_i + (3av - b)y_i^2 + ay_i^3)dt + N\kappa \cdot \frac{1}{N}\sum_{\substack{j=1\\j\neq i}}^{N}(x_{\tau,j} - x_i)dt + \sqrt{2d}\, dw_i, \tag{S4b}$$

that is

$$dx_i = y_i dt, \tag{S5a}$$

$$dy_i = -((3av^2 - 2bv + c)y_i + (3av - b)y_i^2 + ay_i^3)dt + k(\langle x_\tau\rangle - x_i)dt + \sqrt{2d}\, dw_i, \tag{S5b}$$

where

$$k = \lim_{\substack{N\to\infty\\ \kappa\to 0}} N\kappa. \tag{S6}$$

We will denote all mean field quantities q as

$$\langle q \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} q_j, \tag{S7}$$

and for each block element we will introduce deviations from the mean fields

$$n_{x,i} = x_i - \langle x \rangle \,, \qquad n_{y,i} = y_i - \langle y \rangle \,. \tag{S8}$$

We assume that these fluctuations are Gaussian and statistically independent in the different elements.

There is a set of moments known among physicists as cumulants (Lax et al. (2006), Gardiner (1997)) or among statisticians as Thiele semi-invariants, which have the important property that they all vanish in the Gaussian case. We will introduce the following notation for the first and second order cumulants:

- the means — $m_x = \langle x \rangle$, $m_y = \langle y \rangle$,
- the variances — $s_x = \langle n_x^2 \rangle$, $s_y = \langle n_y^2 \rangle$,
- and the cross-cumulant — $u = \langle n_x n_y \rangle$.

From the cumulant analysis, we can determine that

$$\langle y^2 \rangle = s_y + m_y^2, \qquad \langle y^3 \rangle = m_y^3 + 3 m_y s_y, \qquad \langle xy \rangle = u + m_x m_y. \tag{S9}$$

If we apply this to (S5b) and note that the averaging removes the noise term, we get

$$\frac{dm_x}{dt} = m_y, \tag{S10a}$$

$$\frac{dm_y}{dt} = -(3av^2 - 2bv + c)m_y - (3av - b)m_y^2 - am_y^3 \tag{S10b}$$
$$- (3av - b)s_y - 3am_y s_y + k(m_{x,\tau} - m_x).$$

Using the definitions of s_x, s_y and u, (S9) and Itô's chain rule (which is responsible for the reintroduction of d-dependence), it is straightforward, if somewhat laborious, to derive

$$\frac{ds_x}{dt} = 2u; \tag{S11a}$$

$$\frac{ds_y}{dt} = -2s_y[(3av^2 - 2bv + c) + 2(3av - b)m_y + 3a(m_y^2 + s_y)] - 2ku + 2d, \tag{S11b}$$

$$\frac{du}{dt} = -u[(3av^2 - 2bv + c) + 2(3av - b)m_y + 3a(m_y^2 + s_y)] - ks_x + s_y. \tag{S11c}$$

Together, the equations (S10a,b) and (S11a–c) form the system (6), which governs the time evolution of our model:

$$\frac{dm_x}{dt} = m_y, \tag{S12a}$$

$$\frac{dm_y}{dt} = -(3av^2 - 2bv + c)m_y - (3av - b)m_y^2 - am_y^3 \\ - (3av - b)s_y - 3am_y s_y + k(m_{x,\tau} - m_x), \tag{S12b}$$

$$\frac{ds_x}{dt} = 2u; \tag{S12c}$$

$$\frac{ds_y}{dt} = -2s_y[(3av^2 - 2bv + c) + 2(3av - b)m_y + 3a(m_y^2 + s_y)] - 2ku + 2d, \tag{S12d}$$

$$\frac{du}{dt} = -u[(3av^2 - 2bv + c) + 2(3av - b)m_y + 3a(m_y^2 + s_y)] - ks_x + s_y. \tag{S12e}$$

REFERENCES

Gardiner, W. P. (1997). *Statistical Analysis Methods for chemists: A software-based approach* (The Royal Society of Chemistry)

Lax, M., Cai, W., and Xu, M. (2006). *Random Processes in Physics and Finance* (Oxford University Press). doi:10.1093/acprof:oso/9780198567769.001.0001